ON THERMONUCLEAR REACTION RATES

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Abstract

Nuclear reactions govern major aspects of the chemical evolution of galaxies and stars. Analytic study of the reaction rates and reaction probability integrals is attempted here. Exact expressions for the reaction rates and reaction probability integrals for nuclear reactions in the cases of nonresonant, modified nonresonant, screened nonresonant and resonant cases are given. These are expressed in terms of H-functions, G-functions and in computable series forms. Computational aspects are also discussed.

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I. Introduction

Nuclear reactions govern major aspects of the chemical evolution of the universe or, at least, its building blocks: galaxies and stars. A proper understanding of the nuclear reactions that are going on in hot cosmic plasmas, and those in the laboratories as well, requires a sound theory of nuclear-reaction dynamics (Brown and Jarmie[1]). The nuclear reaction rate is the central quantity connecting the theoretical models of universal, galactic and stellar evolution and the nucleosynthesizing nuclear reactions, which can be studied to some extent in nuclear-physics laboratories. Compilations of reaction rates and uncertainties (analytic expressions and tabulated temperature-dependent values) and astrophysical S-factors (analytic expressions and tabulated energy-dependent values) for charged-particle induced reactions are available on the World Wide Web (http://csa5.lbl.gov/chu/astro.html). The rate r_{ij} of reacting particles i and j, in the case of nonrelativistic nuclear reactions taking place in a nondegenerate environment, is usually expressed as

$$
\begin{aligned}
r_{ij} &= n_i n_j \left(\frac{8}{\pi\mu}\right)^{\frac{1}{2}} \left(\frac{1}{kT}\right)^{\frac{3}{2}} \int_0^\infty E\sigma(E)\mathrm{e}^{-\frac{E}{kT}}\,\mathrm{d}E \\
&= n_i n_j < \sigma v >
\end{aligned}
\tag{1.1}
$$

where n_i and n_j denote the particle number densities of the reacting particles i and j, $\mu = \frac{m_i m_j}{m_i + m_j}$ is the reduced mass of the reacting particles, T is the temperature, k is the Boltzmann constant, $\sigma(E)$ is the reaction cross section and $v = (2E/\mu)^{\frac{1}{2}}$ is the relative velocity. Thus $< \sigma v >$ is the reaction probability integral, that is, the probability per unit time that two particles, confined to a unit volume, will react with each other. The basic assumptions made in (1.1) are that the reacting particles i and j have isotropic Maxwell-Boltzmann kinetic-energy distributions and that the kinematic of the reaction can be treated in the center-of-mass system, see Fowler[2], Mathai and Haubold[3], and Imshennik[4].

For nonresonant nuclear reactions between nuclei of charges z_i and z_j at low energies (below the Coulomb barrier), the reaction cross section has the form

$$\sigma(E) = \frac{S(E)}{E}\mathrm{e}^{-2\pi\eta(E)}$$

with

$$\eta(E) = \left(\frac{\mu}{2}\right)^{\frac{1}{2}} \frac{z_i z_j e^2}{\hbar E^{1/2}}$$

where $\eta(E)$ is the Sommerfeld parameter, $\hbar$ is Planck's quantum of action and e is the quantum of electric charge. For a slowly varying cross-section factor $S(E)$ we may expand

$$S(E) = S(0) + \frac{\mathrm{d}S(0)}{\mathrm{d}E}E + \frac{1}{2}\frac{\mathrm{d}^2 S(0)}{\mathrm{d}E^2}E^2.$$

Thus

$$
\begin{aligned}
< \sigma v > = \left(\frac{8}{\pi\mu}\right)^{\frac{1}{2}} \sum_{\nu=0}^{2} \frac{1}{(kT)^{-\nu+\frac{1}{2}}} \frac{S^{(\nu)}(0)}{\nu!} \\
\times \int_0^\infty y^\nu \mathrm{e}^{-y}\mathrm{e}^{-zy^{-\frac{1}{2}}}\,\mathrm{d}y
\end{aligned}
$$

where

$$y = \frac{E}{kT} \text{ and } z = 2\pi\left(\frac{\mu}{2kT}\right)^{\frac{1}{2}} \frac{z_i z_j e^2}{\hbar}.$$

Then the integral to be evaluated is

$$N_\nu(z) = \int_0^\infty y^\nu \mathrm{e}^{-y}\mathrm{e}^{-zy^{-\frac{1}{2}}}\,\mathrm{d}y. \tag{1.2}$$

We will consider a general integral of the following form:

Nonresonant case:

$$I_1 = \int_0^\infty y^\nu \mathrm{e}^{-ay} \mathrm{e}^{-zy^{-\rho}} \mathrm{d}y,\ a > 0,\ z > 0,\ \rho > 0. \tag{1.3}$$

It may be a stringent assumption to consider a thermonuclear fusion plasma as being in thermodynamic equilibrium. If there is a cut-off of the high energy tail of the Maxwell-Boltzmann distribution then (1.2) becomes the following:

Nonresonant case with high energy cut-off:

$$I_2^d = \int_0^d y^\nu \mathrm{e}^{-ay} \mathrm{e}^{-zy^{-\rho}},\ d < \infty,\ a > 0,\ z > 0,\ \rho > 0. \tag{1.4}$$

We consider also ad hoc modifications of the Maxwell-Boltzmann distribution for the evaluation of the nonresonant thermonuclear reaction rate, which acts as a depletion of the tail of the distribution function (see Kaniadakis et al.[5] In this case the general integral to be evaluated is the following:

Nonresonant case with depleted tail:

$$I_3 = \int_0^\infty y^\nu \mathrm{e}^{-ay} \mathrm{e}^{-by^\delta} \mathrm{e}^{-zy^{-\rho}} \mathrm{d}y,\ \delta > 0,\ z > 0,\ \rho > 0,\ a > 0.\ b > 0. \tag{1.5}$$

Taking into account the electron screening effects for the reacting particles, Haubold and Mathai[6] consider the case of the screened nonresonant nuclear reaction rate (see Lapenta and Quarati[7]). In this case

$$\begin{aligned} r_{ij} &= n_i n_j \left(\frac{8}{\pi\mu}\right)^{\frac{1}{2}} \sum_{\nu=0}^{2} \frac{1}{(kT)^{-\nu+\frac{1}{2}}} \frac{S^{(\nu)}(0)}{\nu!} \\ &\times \int_0^\infty y^\nu \mathrm{e}^{-y} \mathrm{e}^{-z\left(y+\frac{b}{kT}\right)^{-\frac{1}{2}}} \mathrm{d}y. \end{aligned} \tag{1.6}$$

In this case we consider the following general integral:

Screened nonresonant case:

$$I_4 = \int_0^\infty y^\nu \mathrm{e}^{-ay} \mathrm{e}^{-z(y+t)^{-\rho}} \mathrm{d}y,\ t > 0,\ \rho > 0,\ z > 0,\ a > 0. \tag{1.7}$$

When the nuclear cross section $\sigma(E)$ in (1.1) has a broad single resonance it can be expressed via the parametrized Breit-Wigner formula and then we have, see also Haubold and Mathai[8],

$$\begin{aligned} < \sigma v > &= (2\pi)^{\frac{5}{2}} \frac{z_i z_j e^2 R_0 w \Gamma_{kl} D}{\mu^{\frac{1}{2}} (kT)^{\frac{3}{2}}} \\ &\times \frac{1}{1 + \left(\frac{1}{2}\Gamma_1\right)^2} \int_0^\infty \frac{\mathrm{e}^{-ay - qy^{-\frac{1}{2}}}}{(b-y)^2 + g^2} \mathrm{d}y \end{aligned} \tag{1.8}$$

where

$$a = \frac{1}{kT\left(1 + \left(\frac{1}{2}\Gamma_1\right)^2\right)}, \quad b = E_r - \frac{1}{4}\Gamma_0\Gamma_1,$$

$$g = \frac{1}{2}(\Gamma_0 + E_r\Gamma_1), \quad q = \bar{q}\left(1 + \left(\frac{1}{2}\Gamma_1\right)^2\right)^{\frac{1}{2}},$$

$$\bar{q} = 2\pi\left(\frac{\mu}{2}\right)^{\frac{1}{2}} \frac{z_i z_j e^2}{h} = a(kT)^{\frac{1}{2}}.$$

Thus the general integral to be evaluated in this case is of the following form:

Resonant case:

$$I_5 = \int_0^\infty \frac{y^\nu \mathrm{e}^{-ay-zy^{-\rho}}}{(b-y)^2+g^2}\mathrm{d}y. \tag{1.9}$$

In the resonant case also we can consider a modification of the Maxwell-Boltzmann distribution which results in a depletion of the tail. Then the general integral to be evaluated is the following:

Resonant case with the depleted tail:

$$I_6 = \int_0^\infty \frac{y^\nu \mathrm{e}^{-ay-by^\delta-zy^{-\rho}}}{(c-y)^2+g^2}\mathrm{d}y. \tag{1.10}$$

The aim of the present article is to give new exact analytic representations of the integrals in (1.2) to (1.10). Additional representations are available from Mathai and Haubold[3].

II. Reduction formulae for the reaction probability integrals

Writing

$$I_2^{(d)} = I_2^{(d)}(\nu, a, z, \rho)$$

we have

$$I_2^{(\infty)} = I_1.$$

Now consider

$$\begin{aligned} I_3 &= \int_0^\infty y^\nu \mathrm{e}^{-ay}\mathrm{e}^{-by^\delta}\mathrm{e}^{-zy^{-\rho}}\mathrm{d}y \\ &= \sum_{m=0}^\infty \frac{(-b)^m}{m!}\int_0^\infty y^{\nu+\delta m}\mathrm{e}^{-ay}\mathrm{e}^{-zy^{-\rho}}\mathrm{d}y \\ &= \sum_{m=0}^\infty \frac{(-b)^m}{m!} I_2^{(\infty)}(\nu+\delta m, a, z, \rho) \end{aligned} \tag{2.1}$$

where $(a)_n$ denotes the Pochhammer symbol,

$$(a)_n = a(a+1)...(a+n-1),\ (a)_0 = 1,\ a \neq 0.$$

Note that

$$\begin{aligned} I_4 &= \int_0^\infty y^\nu \mathrm{e}^{-ay}\mathrm{e}^{-z(y+t)^{-\rho}}\mathrm{d}y \\ &= \mathrm{e}^{at}\int_t^\infty (u-t)^\nu \mathrm{e}^{-au}\mathrm{e}^{-zu^{-\rho}}\mathrm{d}u \\ &= \mathrm{e}^{at}\sum_{m=0}^\infty \frac{(-\nu)_m}{m!}t^m \int_{u=t}^\infty u^{-m}\mathrm{e}^{-au}\mathrm{e}^{-zu^{-\rho}}\mathrm{d}u \\ &= \mathrm{e}^{at}\sum_{m=0}^\infty \frac{(-\nu)_m}{m!}t^m \left[I_2^{(\infty)}(-m, a, z, \rho) - I_2^{(t)}(-m, a, z, \rho)\right]. \end{aligned} \tag{2.2}$$

For simplifying I_5 and I_6 we will use the identity

$$\frac{1}{(c-y)^2+g^2} = \int_0^\infty \mathrm{e}^{-[(c-y)^2+g^2]x}\mathrm{d}x$$

and rewrite the single integral as a double integral. That is,

$$
\begin{aligned}
I_5 &= \int_0^\infty \frac{y^\nu \mathrm{e}^{-ay}\mathrm{e}^{-zy^{-\rho}}}{(c-y)^2+g^2}\mathrm{d}y \\
&= \int_{x=0}^\infty \mathrm{e}^{-g^2x}\int_{y=0}^\infty y^\nu \mathrm{e}^{-x(c-y)^2}\mathrm{e}^{-ay}\mathrm{e}^{-zy^{-\rho}}\mathrm{d}y\mathrm{d}x.
\end{aligned}
$$

Now we expand

$$
\begin{aligned}
\mathrm{e}^{-x(c-y)^2} &= \sum_{m=0}^\infty \frac{(-1)^m(c-y)^{2m}x^m}{m!} \\
&= \sum_{m=0}^\infty \sum_{m_1=0}^{2m} \binom{2m}{m_1}(-1)^{m+m_1}\frac{c^{2m-m_1}}{m!}x^m y^{m_1}.
\end{aligned}
$$

The integral over x gives

$$
\int_{x=0}^\infty x^m \mathrm{e}^{-g^2x}\mathrm{d}x = (g^2)^{-(m+1)}m!. \tag{2.3}
$$

Substituting back we have

$$
\begin{aligned}
I_5 &= \frac{1}{g^2}\sum_{m=0}^\infty \sum_{m_1=0}^{2m} \frac{(2m)!c^{2m-m_1}}{m_1!(2m-m_1)!}\frac{(-1)^{m+m_1}}{(g^2)^m} \\
&\times \int_0^\infty y^{\nu+m_1}\mathrm{e}^{-ay}\mathrm{e}^{-zy^{-\rho}}\mathrm{d}y \\
&= \frac{1}{g^2}\sum_{m=0}^\infty \sum_{m_1=0}^{2m} \binom{2m}{m_1}\frac{(-1)^{m_1}}{c^{m_1}}\left(-\frac{c^2}{g^2}\right)^m I_2^{(\infty)}(\nu+m_1,a,z,\rho).
\end{aligned} \tag{2.4}
$$

Thus it is seen that all the integrals I_1 to I_6 can be reduced to the integral $I_2^{(d)}(\nu,a,z,\rho)$ for two different situations of non-negative as well as negative ν. We will evaluate $I_2^{(d)}$ in the next section by using a statistical technique.

III. Evaluation of the integral $I_2^{(d)}$

In general, integrals I_1 to I_6 are quite difficult to evaluate analytically. Here we will use a statistical technique. We will evaluate the density of a product of two independently distributed real scalar random variables by using two different methods, one procedure leading to the integral that we want to evaluate and the other procedure leading to a representation in terms of a known function. Then appealing to the uniqueness of the density we have the integral evaluated in terms of a known special function. Let x and y be real scalar random variables having the densities

$$
f_1(x) = \begin{cases} c_1\mathrm{e}^{-ax}, & 0<x<d \\ 0, & \text{elsewhere} \end{cases}
$$

and

$$
f_2(y) = \begin{cases} c_2y^\nu\mathrm{e}^{-zy^\rho}, & 0<y<\infty \\ 0, & \text{elsewhere} \end{cases}
$$

where c_1 and c_2 are normalizing factors such that

$$
\int_{x=0}^d f_1(x)\mathrm{d}x = 1 \text{ and } \int_{y=0}^\infty f_2(y)\mathrm{d}y = 1.
$$

Since the variables are assumed to be independently distributed, the joint density of x and y is the product of $f_1(x)$ and $f_2(y)$. Let us transform x and y to $u = xy$ and $v = x$. Then the joint density of u and v, denoted by $g(u,v)$, and the marginal density of u, denoted by $g_1(u)$, are given by

$$g(u,v) = c_1 c_2 u^{\nu} v^{-\nu-1} \mathrm{e}^{-av} \mathrm{e}^{-cv^{-\rho}},\ c = zu^{\rho}$$

and

$$g_1(u) = c_1 c_2 u^{\nu} \int_0^d v^{-\nu-1} \mathrm{e}^{-av} \mathrm{e}^{-cv^{-\rho}} \mathrm{d}v,\ c = zu^{\rho}.$$

Hence we have

$$\int_0^d v^{-\nu-1} \mathrm{e}^{-av} \mathrm{e}^{-cv^{-\rho}} \mathrm{d}v = \frac{u^{-\nu}}{c_1 c_2} g_1(u),\ c = zu^{\rho}. \tag{3.1}$$

Let us look at the $(s-1)$-th moment of u. Due to independence

$$E(u^{s-1}) = [E(x^{s-1})][E(y^{s-1})].$$

But

$$\begin{aligned} E(x^{s-1}) &= c_1 \int_0^d x^{s-1} \mathrm{e}^{-ax} \mathrm{d}x \\ &= c_1 \sum_{m=0}^{\infty} \frac{(-1)^m (ad)^m}{m!} \frac{d^s}{s+m}, \quad \text{for} \quad d < \infty \\ &= c_1 a^{-s} \Gamma(s), \Re(s) > 0, \quad \text{for} \quad d = \infty \end{aligned}$$

where $\Re(\cdot)$ denotes the real par of $(\cdot)$, and

$$\begin{aligned} E(y^{s-1}) &= c_2 \int_0^{\infty} y^{\nu+s-1} \mathrm{e}^{-zy^{\rho}} \mathrm{d}y \\ &= \frac{c_2}{\rho} z^{-\frac{\nu+s}{\rho}} \Gamma\left(\frac{\nu+s}{\rho}\right), \quad \text{for } \Re(\nu+s) > 0. \end{aligned}$$

Taking the inverse Mellin transform of $E(u^{s-1})$ we have

$$\begin{aligned} g_1(u) = &\frac{c_1 c_2}{\rho} z^{-\frac{\nu}{\rho}} \sum_{m=0}^{\infty} \frac{(-1)^m (ad)^m}{m!} \\ &\times \frac{1}{2\pi i} \int_L \frac{d^s}{s+m} z^{-\frac{s}{\rho}} \Gamma\left(\frac{\nu+s}{\rho}\right) u^{-s} \mathrm{d}s \end{aligned} \tag{3.2}$$

where L is a suitable contour and $i = \sqrt{-1}$. This contour integral can be written as an H-function, see for example Mathai and Saxena[9]. That is,

$$\frac{1}{2\pi i} \int_L \frac{d^s}{s+m} z^{-\frac{s}{\rho}} \Gamma\left(\frac{\nu+s}{\rho}\right) u^{-s} \mathrm{d}s = H_{1,2}^{2,0}\left[\frac{uz^{\frac{1}{\rho}}}{d}\middle|_{(m,1),(\frac{\nu}{\rho},\frac{1}{\rho})}^{(m+1,1)}\right]. \tag{3.3}$$

Substituting (3.3) in (3.2) and then comparing with (3.1) we have

$$\int_0^d v^{-\nu-1} \mathrm{e}^{-av} \mathrm{e}^{-zu^{\rho} v^{-\rho}} \mathrm{d}v = I_2^{(d)}(-\nu-1, a, zu^{\rho}, \rho) \tag{3.4}$$

$$\begin{aligned} &= \frac{z^{-\frac{\nu}{\rho}} u^{-\nu}}{\rho} \sum_{m=0}^{\infty} \frac{(-ad)^m}{m!} \\ &\times H_{1,2}^{2,0}\left[\frac{uz^{\frac{1}{\rho}}}{d}\middle|_{(m,1),(\frac{\nu}{\rho},\frac{1}{\rho})}^{(m+1,1)}\right], \quad \text{for } d < \infty \end{aligned} \tag{3.5}$$

$$= \frac{z^{-\frac{\nu}{\rho}} u^{-\nu}}{\rho} H_{0,2}^{2,0}\left[uaz^{\frac{1}{\rho}}\middle|_{(0,1),(\frac{\nu}{\rho},\frac{1}{\rho})}\right], \quad \text{for } d = \infty \tag{3.6}$$

where $\Re(\nu) > 0, \Re(a) > 0, \Re(z) > 0, \Re(\rho) > 0, 0 \leq u \leq \infty$. Note that (3.4) , (3.5) and (3.6) give an $I_2^{(d)}(\nu, a, z, \rho)$ with ν negative. If the integral for a positive ν is required then we proceed as follows: Take $f_1(x) = c_3 x^\nu \mathrm{e}^{-ax}$ and $f_2(y) = c_4 \mathrm{e}^{-zx^\rho}$, where c_3 and c_4 are the new normalizing constants, and then proceed as before. Then we end up with the following representations.

$$\int_0^d v^{\nu-1} \mathrm{e}^{-av} \mathrm{e}^{-zv^{-\rho}} \mathrm{d}v = I_2^{(d)}(\nu - 1, a, z, \rho) \tag{3.7}$$

$$= \frac{d^\nu}{\rho} \sum_{m=0}^{\infty} \frac{(-ad)^m}{m!}$$

$$\times H_{1,2}^{2,0}\left[\frac{z^{\frac{1}{\rho}}}{d}\middle| \begin{matrix} (\nu+m+1,1) & \\ (\nu+m,1), & \left(0,\frac{1}{\rho}\right)\end{matrix}\right] \text{ for } d < \infty \tag{3.8}$$

$$= \frac{a^{-\nu}}{\rho} H_{0,2}^{2,0}\left[az^{\frac{1}{\rho}}|_{(\nu,1),\left(0,\frac{1}{\rho}\right)}\right], \text{ for } d = \infty \tag{3.9}$$

where $\Re(\nu) > 0,\ \Re(a) > 0,\ \Re(z) > 0,\ \Re(\rho) > 0$. Thus (3.7), (3.8) and (3.9) cover the case of a positive ν in $I_2^{(d)}(\cdot)$. When ρ is real and rational then the H-functions appearing in (3.3) to (3.9) can be reduced to G-functions by using the multiplication formula for gamma functions, namely,

$$\Gamma(mz) = (2\pi)^{\frac{(1-m)}{2}} m^{mz-\frac{1}{2}} \Gamma(z)\Gamma\left(z + \frac{1}{m}\right) ... \Gamma\left(z + \frac{m-1}{m}\right),\ m = 1, 2, ... \tag{3.10}$$

For the theory and applications of G-functions see for example Mathai[1]0. For $\rho = \frac{m}{n},\ m, n = 1, 2, ...$ reduction to the G-function is available from Mathai and Haubold[3].

The parameters of interest in nuclear astrophysics are $I_2^{(d)}(\nu, a, z, \rho)$ for $a = 1,\ z > 0,\ \rho = \frac{1}{2}$. In this case computable representations will be discussed in the next section.

IV. Computable series representations of the reaction rate integrals

Let us start with (3.9) for $\rho = \frac{1}{2}$. Then the H-function to be evaluated is the following:

$$H_{0,2}^{2,0}(\cdot) = \frac{1}{2\pi i} \int \Gamma(\nu + s)\Gamma(2s)(az^2)^{-s} \mathrm{d}s$$

$$= \frac{1}{2\sqrt{\pi}} \frac{1}{2\pi i} \int \Gamma(s)\Gamma\left(s + \frac{1}{2}\right)\Gamma(\nu + s)\left(\frac{az^2}{4}\right)^{-s} \mathrm{d}s \tag{4.1}$$

by expanding $\Gamma(2s)$ with the help of (3.10). Note that (4.1) is a G-function of the type $G_{0,p}^{p,0}(\cdot)$, see for example Mathai[10]. Observe that for $\nu \neq \pm\frac{\lambda}{2},\ \lambda = 0, 1, ...$ all the poles of the integrand are simple. Then (4.1) generates a series corresponding to each gamma in the integrand. Corresponding to $\Gamma(s)$ the poles are at $s = -n,\ n = 0, 1, ...$ and the corresponding residue is

$$\lim_{s \to -n}\left[(s+n)\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)\Gamma(\nu+s)\left(\frac{az^2}{4}\right)^{-s}\right] = \frac{(-1)^n}{n!}\Gamma\left(-n+\frac{1}{2}\right)\Gamma(\nu-n)\left(\frac{az^2}{4}\right)^n.$$

But

$$\Gamma\left(-n+\frac{1}{2}\right) = \frac{(-1)^n \Gamma\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)_n}$$

and

$$\Gamma(\nu - n) = \frac{(-1)^n \Gamma(\nu)}{(1-\nu)_n}.$$

Hence the sum of the residues gives

$$\Gamma\left(\frac{1}{2}\right)\Gamma(\nu)\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{az^2}{4}\right)^n}{\left(\frac{1}{2}\right)_n(1-\nu)_n}=\Gamma\left(\frac{1}{2}\right)\Gamma(\nu){}_0F_2\left(\ ;\frac{1}{2},1-\nu;-\frac{az^2}{4}\right) \tag{4.2}$$

where ${}_0F_2(\cdot)$ is a hypergeometric series which is convergent for all a and b. Thus (3.9) is a linear function of 3 such series of ${}_0F_2$'s for $\nu \neq \pm\frac{n}{2},\ n=0,1,....$ If ν is an integer or half-integer then we have one set of poles of order 2 each and one set of order one each. Techniques for handling the integral when the integrand has higher order poles are given in Mathai[10]. Series representations of (4.1) for all cases of ν are given in Mathai and Haubold[3].

Now let us examine (3.8) for $\rho=\frac{1}{2}$. In this case the H-function to be evaluated is given by

$$\begin{aligned} H &= H_{1,2}^{2,0}\left[\frac{z^2}{d}\Big|_{(\nu+m,1),(0,2)}^{(\nu+m+1,1)}\right] \\ &= \frac{1}{2\pi i}\int\frac{1}{\nu+m+s}\Gamma(2s)\left(\frac{z^2}{d}\right)^{-s}\mathrm{d}s \\ &= \frac{1}{2\pi i}\int\frac{1}{\nu+m+s}\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)\left(\frac{z^2}{d}\right)^{-s}\mathrm{d}s. \end{aligned} \tag{4.3}$$

At $s=-\nu-m$ there is a pole of order one if $\nu\neq\pm\frac{\lambda}{2},\ \lambda=0,1,...$, otherwise it will be a pole of order 2 at this point. When it is a pole of order one the residue is given by

$$\begin{aligned} &\frac{1}{2\sqrt{\pi}}\Gamma(-\nu-m)\Gamma\left(-\nu-m+\frac{1}{2}\right)\left(\frac{z^2}{d}\right)^{m+\nu} \\ &\quad=\frac{1}{2\sqrt{\pi}}\frac{\Gamma(-\nu)\Gamma\left(-\nu-\frac{1}{2}\right)}{(\nu+1)_m\left(\nu+\frac{1}{2}\right)_m}\left(\frac{z^2}{d}\right)^{m+\nu}. \end{aligned}$$

Hence corresponding to this residue the term in (3.8) is the following:

$$\begin{aligned} &\frac{z^{2\nu}}{\sqrt{\pi}}\Gamma(-\nu)\Gamma\left(-\nu+\frac{1}{2}\right)\sum_{m=0}^{\infty}\frac{1}{(\nu+1)_m\left(\nu+\frac{1}{2}\right)_m}(-az^2)^m \\ &\quad=\frac{z^{2\nu}}{\sqrt{\pi}}\Gamma(-\nu)\Gamma\left(-\nu+\frac{1}{2}\right){}_0F_2\left(\ ;\nu+1,\nu+\frac{1}{2};-az^2\right) \end{aligned}$$

which is convergent.

At $s=-n,\ n=0,1,...$ the integrand in (4.3) has poles of order one when $\nu\neq\pm\frac{\lambda}{2},\ \lambda=0,1,...$ The sum of the residues here is given by

$$\begin{aligned} &\frac{1}{2\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{1}{\nu+m-n}\frac{(-1)^n}{n!}\Gamma\left(-n+\frac{1}{2}\right)\left(\frac{z^2}{d}\right)^n \\ &\quad=\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{\nu+m-n}\frac{1}{\left(\frac{1}{2}\right)_n n!}\left(\frac{z^2}{d}\right)^n. \end{aligned} \tag{4.4}$$

The term corresponding to this in (3.8) is the following double series:

$$d^{\nu}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-ad)^m}{m!}\frac{1}{\nu+m-n}\frac{1}{\left(\frac{1}{2}\right)_n}\frac{(z^2/d)^n}{n!}.$$

By Horn's theorem on convergence, see for example Srivastava and Karlsson[11] this double series is convergent for all a, z and d. The third term of (3.8) as well as all terms for other cases of ν can be seen to give convergent series for all $0 < d < \infty,\ a > 0,\ z > 0$. Similar arguments hold good for (3.5) and (3.6) also.

Let us examine I_3. We have expressed I_3 in terms of $I_2^{(\infty)}$ in (2.1). That is,

$$I_3 = \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} I_2^{(\infty)}(\nu + \delta m, a, z, \rho).$$

For $\rho = \frac{1}{2}$ a typical term in this $I_2^{(\infty)}$ behaves like a ${}_0F_2$ given in (4.2). A typical term will be a constant multiple of a double series of the form

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-b)^m}{m!} \frac{1}{\left(\frac{1}{2}\right)_n (1-\nu-\delta m)_n} \frac{\left(-\frac{az^2}{4}\right)^n}{n!}. \tag{4.4}$$

This by Horn's theorem is convergent for all $(x, y)|0 < x < \infty,\ 0 < y < \infty$ where $x = b$ and $y = \frac{az^2}{4}$. Hence the series representation of I_3 is convergent for all a, b, z and δ for $\nu \neq \pm\frac{\lambda}{2},\ \lambda = 0, 1, \ldots$. When ν is an integer or half-integer then the series representation will contain psi functions and logarithmic terms but from the structure in (4.4) one can see that the series will be convergent.

Now let us examine the complicated form coming from I_5 or from (2.4). Here we have the factor $I_2^{(\infty)}(\nu + m_1, a, z, \rho)$ with $m_1 \geq 0$. For the case $\rho = \frac{1}{2}$ we have from (3.9) and (4.1)

$$\begin{aligned}\int_0^{\infty} x^{\nu+m_1} \mathrm{e}^{-ax} \mathrm{e}^{-zx^{-\frac{1}{2}}} \mathrm{d}x &= I_2^{(\infty)}\left(\nu + m_1, a, z, \frac{1}{2}\right) \\ &= \frac{a^{-(\nu+m_1+1)}}{\sqrt{\pi}} \frac{1}{2\pi i}\int_L \Gamma(s)\Gamma\left(s+\frac{1}{2}\right)\Gamma(\nu+m_1+1+s)\left(\frac{az^2}{4}\right)^{-s}\mathrm{d}s. \end{aligned}\tag{4.5}$$

Writing (4.5) as a G-function, see Mathai[10], we have

$$I_2^{(\infty)}\left(\nu + m_1, a, b, \frac{1}{2}\right) = \frac{a^{-(\nu+m_1+1)}}{\sqrt{\pi}} G_{0,3}^{3,0}\left[\frac{az^2}{4}\Big|_{0,\frac{1}{2},\nu+m_1+1}\right]. \tag{4.6}$$

The behavior of this G-function for small and large values of $\frac{az^2}{4}$ is available from Mathai and Saxena[12] or Luke[13]. For small values of $\frac{az^2}{4}$ this G-function behaves like unity and hence $I_2^{(\infty)}$ behaves like $\frac{a^{-(\nu+m_1)}}{\sqrt{\pi}}$. For large values of $\frac{az^2}{4}$ the $I_2^{(\infty)}$ in (4.6) behaves like

$$\left(\frac{az^2}{4}\right)^{\frac{1}{6}}\left(\frac{az^2}{4}\right)^{\frac{\nu+m_1}{3}} \mathrm{e}^{-3\left(\frac{az^2}{4}\right)^{\frac{1}{3}}}.$$

Hence for checking the convergence of I_5 in (2.4) it is sufficient to examine the two series

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{m_1=0}^{2m}\binom{2m}{m_1}\left(-\frac{1}{c}\right)^{m_1} a^{-m_1}\left(-\frac{c^2}{g^2}\right)^m \\ &\quad = \sum_{m=0}^{\infty}\left[\left(1-\frac{1}{ca}\right)^2\right]^m\left(-\frac{c^2}{g^2}\right)^m \\ &\quad = \left[1+\frac{c^2}{g^2}\left(1-\frac{1}{ca}\right)^2\right]^{-1} \quad \text{for} \quad \left|\frac{c}{g}\left(1-\frac{1}{ca}\right)\right| < 1\end{aligned}$$

or for $\frac{1}{a}-|g|<c<\frac{1}{a}+|g|$ which is equivalent to $0<c<1+|g|$ for $|g|\geq 1$, when $a=1$, and

$$\sum_{m=0}^{\infty}\sum_{m_1=0}^{2m}\binom{2m}{m_1}\left(-\frac{1}{ac}\right)^{m_1}\left[\left(\frac{az^2}{4}\right)^{\frac{1}{3}}\right]^{m_1}\left(-\frac{c^2}{g^2}\right)^{m}$$
$$=\sum_{m=0}^{\infty}\eta^m\left(-\frac{c^2}{g^2}\right)^m=\left[1+\eta\frac{c^2}{g^2}\right]^{-1},$$

where

$$\eta=\left[1-\frac{1}{c}\left(\frac{z}{2a}\right)^{\frac{2}{3}}\right]^2$$

for $\left|\eta\frac{c^2}{g^2}\right|<1$ or for

$$2a[c-|g|]^{\frac{3}{2}}<z<2a[c+|g|]^{\frac{3}{2}}.$$

For example, combined with the condition for small values, we have $0<z<2(2+|g|)$ for $a=1,\ |g|\geq 2$.

For other values of ρ also the procedure remains the same. We may observe that instead of the I_5 in (1.9) we can also consider a more general integral of the type

$$I_7=\int_0^{\infty}\frac{y^{\nu}\mathrm{e}^{-ay-zy^{-\rho}}}{[(c-y)^2+g^2]^d}\mathrm{d}y,\ \nu,\ a,\ z,\ c,\ \rho,\ d\ >0. \tag{4.7}$$

In this case replace

$$\frac{1}{[(c-y)^2+g^2]^d}=\frac{1}{\Gamma(d)}\int_0^{\infty}x^{d-1}\mathrm{e}^{-x[(c-y)^2+g^2]}\mathrm{d}x,\ \Re(d)>0. \tag{4.8}$$

Then the modification in (2.3) becomes

$$\int_0^{\infty}x^{m+d-1}\mathrm{e}^{-g^2x}\mathrm{d}x=(g^2)^{-(m+d)}\Gamma(m+d) \tag{4.9}$$

and a corresponding expression for (2.4) is obtained. Convergence conditions can be checked exactly the same way as in the case of $d-1=0$.

I_6 can be reduced to a form corresponding to (2.4). Convergence conditions can be checked by converting the series form into one of the standard triple series discussed in Srivastava and Karlsson[11] or by using the general procedure for a multiple series or by writing the kernel function as a G-function, as described above, and then checking the behavior for large and small values of the argument of this G-function. Note that I_7 can be expressed in terms of $I_2^{(\infty)}$ as follows:

$$I_7=\sum_{m=0}^{\infty}(-1)^m\frac{(d)_m}{m!}(g^2)^{-(m+d)}\sum_{m_1=0}^{2m}\binom{2m}{m_1}(-1)^{m_1}c^{2m-m_1}$$
$$\times\sum_{n=0}^{\infty}\frac{(-b)^n}{n!}I_2^{(\infty)}(\nu+m_1+\delta n,a,z,\rho). \tag{4.10}$$

Writing $I_2^{(\infty)}$ in terms of a G-function and then checking the behavior of the G-function for small and large values of the argument one can verify the existence of I_7. For small values, the G-function in I_2 behaves like unity and then for $\rho=\frac{1}{2}$

$$I_7=\sum_{m=0}^{\infty}(-1)^m\frac{(d)_m}{m!}(g^2)^{-(m+d)}\sum_{m_1=0}^{2m}\binom{2m}{m_1}(-1)^{m_1}c^{2m-m_1}$$
$$\times\sum_{n=0}^{\infty}\frac{(-b)^n}{n!}\frac{a^{-(\nu+1+m_1+\delta n)}}{\sqrt{\pi}}$$
$$=\frac{a^{-(\nu+1)}}{\sqrt{\pi}}\mathrm{e}^{-a^{-\delta}}g^{-2d}{}_1F_0\left(d;\ ;-\frac{c^2}{g^2}\left(1-\frac{1}{ca}\right)^2\right)$$

for $\frac{c^2}{g^2}\left[1-\frac{1}{ca}\right]^2<1$ or $\frac{1}{a}-|g|<c<\frac{1}{a}+|g|$. For large values the behavior of the G-function is available from (5.2) later on and in this case the conditions for the existence are given in (5.8) later on.

When $\rho=\frac{m}{n},\ m,n=1,2,...$ it is easy to see that the H-function in all the integrals I_1 to I_6 reduce to G-functions, of course with more parameters. General series representations of all forms of G-functions are available from Mathai[10].

V. Computational aspects

Anderson, Haubold and Mathai[14] looked into the computational aspects of the integrals I_1 to I_4. Exact computations and graphs are given there for $\rho=\frac{1}{2},\ a=1$ in the cases of I_1 for $\nu=1,2$; I_2 for $\nu=0,\ d=1,\ 5$; I_3 for $(\nu,\delta,b)=(0,2,0.001),\ (1,5,1)$; I_4 for $(\nu,t)=(0,1),\ (0,5)$. For large values of z one can use the asymptotic forms of the integrals for computational purposes. These can be worked out by using the asymptotic form of the G-function. From (3.9) and (4.1)

$$I_2^{(\infty)}\left(\nu,a,z,\frac{1}{2}\right)=\frac{a^{-(\nu+1)}}{\sqrt{\pi}}G_{0,3}^{3,0}\left[\frac{az^2}{4}\bigg|_{\nu+1,a,\frac{1}{2}}\right]. \tag{5.1}$$

But for large values of $\frac{az^2}{4}$ we have

$$G_{0,3}^{3,0}\left[\frac{az^2}{4}\bigg|_{\nu+1,0,\frac{1}{2}}\right]\approx\frac{2\sqrt{\pi}}{\sqrt{3}}a^{-(\nu+1)}\mathrm{e}^{-3\left(\frac{az^2}{4}\right)^{\frac{1}{3}}}\left(\frac{az^2}{4}\right)^{\frac{2\nu+1}{6}}. \tag{5.2}$$

By substituting this expression for the G-function in I_1 to I_7 we get the following forms for large values of $\frac{az^2}{4}$ with $a=1,\ \rho=\frac{1}{2}$:

$$I_1\approx 2\left(\frac{\pi}{3}\right)^{\frac{1}{2}}\left(\frac{z^2}{4}\right)^{\frac{2\nu+1}{6}}\mathrm{e}^{-3\left(\frac{z^2}{4}\right)^{\frac{1}{3}}} \tag{5.3}$$

$$I_2\approx d^{\nu+1}\mathrm{e}^{-d}\left(\frac{z^2}{4d}\right)^{-\frac{1}{2}}\mathrm{e}^{-2\left(\frac{z^2}{4d}\right)^{\frac{1}{2}}} \tag{5.4}$$

$$I_3\approx 2\left(\frac{\pi}{3}\right)^{\frac{1}{2}}\left(\frac{z^2}{4}\right)^{\frac{2\nu+1}{6}}\mathrm{e}^{-3\left(\frac{z^2}{4}\right)^{\frac{1}{3}}}\mathrm{e}^{-b\left(\frac{z^2}{4}\right)^{\frac{\delta}{3}}} \tag{5.5}$$

$$I_4\approx 2\left(\frac{\pi}{3}\right)^{\frac{1}{2}}\mathrm{e}^{t}\left(\frac{z^2}{4}\right)^{\frac{1}{6}}\mathrm{e}^{-3\left(\frac{z^2}{4}\right)^{\frac{1}{3}}}\left[\left(\frac{z^2}{4}\right)^{\frac{1}{3}}-t\right]^{\nu} \tag{5.6}$$

$$I_5\approx\frac{2}{g^2}\left(\frac{\pi}{3}\right)^{\frac{1}{2}}\mathrm{e}^{-3\left(\frac{z^2}{4}\right)^{\frac{1}{3}}}\left(\frac{z^2}{4}\right)^{\frac{2\nu+1}{6}}\left[1+\eta\frac{c^2}{g^2}\right]^{-1} \tag{5.7}$$

where $\eta=\left[1-\frac{1}{c}\left(\frac{z}{2}\right)^{\frac{2}{3}}\right]^2$

$$I_6=I_7 \text{ for } d=1$$

$$I_7\approx\frac{2\sqrt{\pi}}{\sqrt{3}}a^{-(\nu+1)}g^{-2d}\left(\frac{az^2}{4}\right)^{\frac{1}{2}+\frac{\nu}{3}}\mathrm{e}^{-3\left(\frac{az^2}{4}\right)^{\frac{1}{3}}}$$
$$\times\,\mathrm{e}^{-\frac{b}{a^\delta}\left(\frac{az^2}{4}\right)^{\frac{\delta}{3}}}{}_1F_0(d;\ ;-\gamma^2),$$

for $|\gamma|<1$ where

$$\gamma^2=\frac{c^2}{g^2}\left[1-\frac{1}{c}\left(\frac{z}{2a}\right)^{\frac{2}{3}}\right]^2.$$

For a broad overview of the aspects of numerical evaluation of special functions, including available software packages for this purpose, see on the World Wide Web: http://math.nist.gov/nest/.

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